A GENERALIZATION OF ALTERNATIVE RINGS(1)

BY

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1. **Introduction.** In their well-known paper [3] Bruck and Kleinfeld proved that any alternative ring must satisfy the identity

$$(x^2, y, z) = x \circ (x, y, z) \tag{1}$$

where the associator (x, y, z) is defined by $(x, y, z) = (xy)z - x(yz)$ and $x \circ y = xy + yx$. By symmetry an alternative ring must also satisfy the dual of the above identity:

$$(z, y, x^2) = x \circ (z, y, x). \tag{2}$$

Let A be a ring satisfying (1) and (2) and suppose further that A has a unit element 1. Then the relations (1) and (2) yield no identities of degree 3 which can be obtained from (1) and (2) by setting one of the variables equal to 1 since for any such substitution the relations (1) and (2) reduce to the trivial equation(2).

In this paper we study the class of rings which satisfy (1), (2), and

$$(x, x, x) = 0. \tag{3}$$

From our earlier remarks it is immediate that these rings are generalizations of alternative rings.

In §2 we show that any ring A satisfying (1), (2), and (3) must be power-associative and, using this result we obtain an idempotent decomposition for A as $A = A_1 + A_{1/2} + A_0$ where $x \in A_i$ if and only if $ex + xe = 2ix$ for the idempotent e of A. In *Theorem* 3 we develop some fundamental relations for the multiplicative properties of the A_i. We are able to show in §3 that if A has no nil ideals then A must, in fact, have a Peirce decomposition with respect to an idempotent e. That is, A is the direct sum of the subgroups A_{ij}; $i, j = 0, 1$ where $x \in A_{ij}$ if and only if $ex = ix, xe = jx$. This is then used to prove the main results: (a) Any simple ring A satisfying (1), (2), and (3) with an idempotent $e \neq 1$ must be associative or a Cayley-Dickson algebra over its center. (b) Any finite-dimensional semi-simple algebra A satisfying (1), (2), and (3) has a unity element and is the direct sum of simple algebras. In §5 we give some examples to show that these results are in a certain sense best possible.

We suppose in the remainder of this paper that the ring A satisfies (1), (2), and (3).

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(2) We refer the reader to [7] for information pertinent to this remark.

2. **Preliminaries.** We begin this section with the following:

THEOREM 1. *A is power-associative.*

Proof. Identity (3) gives $(x, x, x) = 0$ and this along with (1) and (2) yields $x^2x^2 = x^3x = xx^3$. We define x^n inductively by $x^{n-1}x = x^n$. Then we have $x^3 = x^ix^j$ for $i+j = 3, 0 < i, j < 3$ and $x^4 = x^ix^j$ for $i+j = 4$, $0 < i, j < 4$. We now show by induction that $x^n = x^ix^j$ for $i+j = n$, $0 < i, j < n$. We assume that $x^{i+j} = x^ix^j$ for $i+j < n$; $0 < i, j$ and $n \geqq 5$. Then (1) with $y = x^{n-2-i}$, $z = x^i$ becomes $(x^2, x^{n-2-i}, x^i) = x \circ (x, x^{n-2-i}, x^i) = 0$. Thus, $x^{n-i}x^i = x^2x^{n-2}$ for $0 < i < n-2$ so that we have $x^{n-i}x^i = x^n$ except possibly when $i = n-1$. But using $y = x^{n-3}$. $z = x$ in (2) we obtain $(x, x^{n-3}, x^2) = x \circ (x, x^{n-3}, x) = 0$ which gives $x^{n-2}x^2 = xx^{n-1} = x^n$ since $n \geqq 5$. Thus, $x^{i+j} = x^ix^j$ for $0 < i, j$ so that A is power-associative.

Replacing x by $x + w$ in (1) and (2) yields

(1)′ $$(x \circ w, y, z) = x \circ (w, y, z) + w \circ (x, y, z)$$

and

(2)′ $$(z, y, x \circ w) = x \circ (z, y, w) + w \circ (z, y, x).$$

Linearizing (3) leads to the identity

(3)′ $$(x, x, y) + (x, y, x) + (y, x, x) = 0$$

provided that A has characteristic $\neq 2$ and so, whenever necessary, we shall assume in addition that A satisfies (3)′.

Let e be an idempotent of A. Then setting $w = y = z = e$ in (1)′ and (2)′ we find $(xe + ex, e, e) = e \circ (x, e, e)$ and $(e, e, ex + xe) = e \circ (e, e, x)$. In any ring we have $(xe, e, e) = (x, e, e) = (x, e, e)e$ and $(e, e, ex) = e(e, e, x)$ so that the above relations reduce to

(4) $$e(x, e, e) = (ex, e, e), \quad (e, e, xe) = (e, e, x)e.$$

Using the substitutions $x = e, y = x, z = e$ and $x = e, z = x, y = e$ in (1) and (2) respectively we obtain

(5) $$(e, x, e) = e \circ (e, x, e), \quad (e, e, x) = e \circ (e, e, x), \quad (x, e, e) = e \circ (x, e, e).$$

In any ring we have

$$(ex, e, e) - (e, xe, e) + (e, x, e) = e(x, e, e) + (e, x, e)e$$

which with (4) and (5) reduces to $(e, xe, e) = e(e, x, e)$. By symmetry we must also have $(e, ex, e) = (e, x, e)e$. Identities (3)′ and (4) yield:

$$(ex, e, e) + (e, ex, e) + (e, e, ex) = 0 = e(x, e, e) + (e, ex, e) + e(e, e, x).$$

But $e[(x, e, e) + (e, x, e) + (e, e, x)] = 0$ so that $(e, ex, e) = e(e, x, e)$. Thus we have

$$(e, ex, e) = e(e, x, e) = (e, xe, e) = (e, x, e)e. \tag{6}$$

THEOREM 2[3]. *Let e be an idempotent of A. Then $A = A_1 + A_{1/2} + A_0$ where $x \in A_i$; $i = 0,1$ if and only if $ex = xe = ix$, and $x \in A_{1/2}$ if and only if $ex + xe = x$. A is the additive direct sum of the subgroups A_i; $i = 0, 1/2, 1$.*

Proof. Let $x \in A$. We set $x_1 = e(xe) - (e, e, x) = (ex)e + (x, e, e)$ (by (3)′). Then we see that

$$\begin{aligned} ex_1 - x_1 &= e(e(xe)) - e(e, e, x) - e(xe) + (e, e, x) \\ &= -(e, e, xe) - e(e, e, x) + (e, e, x) \\ &= -e \circ (e, e, x) + (e, e, x) = 0 \end{aligned}$$

and

$$\begin{aligned} x_1 e - x_1 &= ((ex)e)e + (x, e, e)e - (ex)e - (x, e, e) \\ &= (ex, e, e) + (x, e, e)e - (x, e, e) \\ &= e \circ (x, e, e) - (x, e, e) = 0. \end{aligned}$$

Hence $ex_1 = x_1 e = x_1$. Next we set $x_0 = x_1 - (ex + xe - x)$ and we see that

$$ex_0 = ex_1 - e(ex + xe - x) = x_1 + (e, e, x) - e(xe) = x_1 - x_1 = 0,$$

$$x_0 e = x_1 e - (ex + xe - x)e = x_1 - (ex)e - (x, e, e) = x_1 - x_1 = 0.$$

Thus $ex_0 = x_0 e = 0$. Finally we set $x_{1/2} = ex + xe - 2x_1$. Then

$$\begin{aligned} ex_{1/2} + x_{1/2}e &= e(ex) + e(xe) + (xe)e + (ex)e - 4x_1 \\ &= -(e,e,x) + (x,e,e) + e(xe) + (ex)e - 4x_1 + ex + xe \\ &= x_1 + x_1 - 4x_1 + ex + xe \\ &= ex + xe - 2x_1 = x_{1/2}. \end{aligned}$$

It is immediate from the definitions of the x_i that $x = x_1 + x_{1/2} + x_0$. This representation of x as the sum of the elements $x_1, x_{1/2}, x_0$ is unique for if $x = x_1 + x_{1/2} + x_0 = 0$ we have $ex + xe = 2x_1 + x_{1/2} = 0$. But then $2x_1 + x_{1/2} - x = x_1 - x_0 = 0$. Thus $e(x_1 - x_0) = x_1 = 0$ so that $x_1 = x_{1/2} = x_0 = 0$. This completes the proof.

Now suppose $x \in A_{1/2}$. Then from (1), $(e, x, e) = e \circ (e, x, e)$ so that $(e,x,e) \in A_{1/2}$. Next let $ex = x_1 + x_{1/2} + x_0$. Then

$$\begin{aligned} (e, e, x) = ex - e(ex) &= x_1 + x_{1/2} + x_0 - x_1 - ex_{1/2} \\ &= e(xe) - x_{1/2} - ex_{1/2} + x_0 = x_{1/2}e + x_0 \end{aligned}$$

and

[3] Except for special characteristics, Theorem 2 and portions of Theorem 3 can be obtained from the results of Albert [1] and Kokoris, *New results on power-associative algebras*, Trans. Amer. Math. Soc. 77 (1954), 363–373.

$$(x, e, e) = (xe)e - xe = -(ex)e = -x_1 - x_{1/2}e.$$

But $(3)'$ implies that $(e, e, x) + (x, e, e) = -(e, x, e) = -x_1 + x_0 \in A_{1/2}$. Hence $x_1 = x_0 = (e, x, e) = 0$ for $x \in A_{1/2}$. Thus $(e, x, e) = 0$ for every $x \in A$. We note that we have also shown above that $ex, xe \in A_{1/2}$ for $x \in A_{1/2}$.

Let us examine some of the multiplicative properties of the A_i. Let $x_1, y_1 \in A_1$. Then substituting $x = x_1$, $w = e$, $y = y_1$, $z = e$ and $x = e$, $y = x_1$, $z = y_1$ in $(1)'$ and (1) respectively we obtain $2(x_1, y_1, e) = e \circ (x_1, y_1, e)$ and $(e, x_1, y_1) = e \circ (e, x_1, y_1)$. Hence

$$(x_1, y_1, e)_{1/2} = (e, x_1, y_1)_1 = (e, x_1, y_1)_0 = 0.$$

In a similar fashion using $(2)'$ and (2) we find

$$(x_1, y_1, e)_1 = (x_1, y_1, e)_0 = (e, x_1, y_1)_{1/2} = 0.$$

Thus $(x_1, y_1, e) = (e, x_1, y_1) = 0$ so that $x_1 y_1 \in A_1$. Replacing x_1, y_1 by $x_0, y_0 \in A_0$ we also find $x_0 y_0 \in A_0$.

Let $x_1 \in A_1$, $y_{1/2} \in A_{1/2}$. Then substituting $x = x_1$, $w = e$, $y = y_{1/2}$, $z = e$ in $(1)'$ we find $2(x_1, y_{1/2}, e) = e \circ (x_1, y_{1/2}, e)$ while setting $x = e$, $y = y_{1/2}$, $z = x_1$ in (2) yields $(x_1, y_{1/2}, e) = e \circ (x_1, y_{1/2}, e)$. Hence $(x_1, y_{1/2}, e) = 0$. Next we set $x = e$, $w = y_{1/2}$, $y = e$, $z = x_1$ in $(2)'$, to obtain $(x_1, e, y_{1/2}) = e \circ (x_1, e, y_{1/2}) \in A_{1/2}$. Then $(x_1, y_{1/2}, e) + (x_1, e, y_{1/2}) = (x_1 y_{1/2})e - x_1(y_{1/2}e) + x_1 y_{1/2} - x_1(e y_{1/2}) = (x_1 y_{1/2})e \in A_{1/2}$. Hence $x_1 y_{1/2} \in A_{1/2} + A_0$. Next we set $x = e$, $y = x_1$, $z = y_{1/2}$ in (1) to obtain $(e, x, y_{1/2}) = e \circ (e, x, y_{1/2})$ so that $x_1 y_{1/2} - e(x_1 y_{1/2}) \in A_{1/2}$. Thus $x_1 y_{1/2} \in (A_1 + A_{1/2}) \cap (A_{1/2} + A_0) = A_{1/2}$.

In a similar fashion $y_{1/2} x_1 \in A_{1/2}$. Replacing x_1 by x_0 we also find that $(x_0, y_{1/2}, e) = (e, y_{1/2}, x_0) = 0$ and $x_0 y_{1/2}$, $y_{1/2} x_0 \in A_{1/2}$. Thus using Albert's terminology [1] every idempotent e of A is stable.

Suppose $x \in A_{1/2}$. Then $(3)'$ yields $ex^2 = x^2 e$. Next using $(1)'$ and $(2)'$ we obtain $(x, e, x) = x \circ (e, e, x) + e \circ (x, e, x)$ and $(x, e, x) = x \circ (x, e, e) + e \circ (x, e, x)$. Thus $x \circ (e, e, x) = x \circ (x, e, e)$. From (1) and (2) we obtain $(x^2, e, e) = x \circ (x, e, e)$ and $(e, e, x^2) = x \circ (e, e, x)$. Hence $(x^2, e, e) = (e, e, x^2)$. Thus

$$\begin{aligned} 0 &= 2(e, e, x^2) - 2(x^2, e, e) = 2ex^2 - 2e(ex^2) - 2(x^2 e)e + 2x^2 e \\ &= 2[2ex^2 - 2e \circ (ex^2)] = 2e(x^2)_{1/2} = (x^2)_{1/2}. \end{aligned}$$

Now let $x_1 \in A_1$, $y_0 \in A_0$. Then $2(x_1, e, y_0) = e \circ (x_1, e, y_0)$ is obtained by setting $x = x_1$, $w = e$, $y = e$, $z = y_0$ in $(1)'$. This reduces to $(x_1 y_0)_{1/2} = 0$. Substituting $x = e$, $y = x_1$, $z = y_0$ in (1) we find $(e, x_1, y_0) = e \circ (e, x_1, y_0)$. Hence $(x_1 y_0)_0 = 0$ and interchanging x_1 and y_0 we find $(y_0 x_1)_0 = 0$. Employing $(2)'$ and (2) we have $(y_0 x_1)_{1/2} = (y_0 x_1)_0 = (x_1 y_0)_1 = 0$. Combining these we have $x_1 y_0 = y_0 x_1 = 0$ and we state

THEOREM 3. *Suppose $A = A_1 + A_{1/2} + A_0$ with respect to the idempotent e of A. Then A_1 and A_0 are orthogonal subrings and $A_iA_{1/2} + A_{1/2}A_i \subseteq A_{1/2}$ for $i = 0,1$. Moreover, the following special relations hold: $(x_i, y_{1/2}, e) = (e, y_{1/2}, x_i) = 0$ for $x_i \in A_i$; $i = 0,1$. If $x_{1/2}, y_{1/2} \in A_{1/2}$ then $x_{1/2}^2 \in A_1 + A_0$ and $(x_{1/2}y_{1/2})_{1/2} = -(y_{1/2}x_{1/2})_{1/2}$.*

3. **Ideals and simple rings.** The following is fundamental in our development.

THEOREM 4. *Let $\mathscr{L} = \{x \mid x \in A_{1/2}$ and $ax, xa \in A_{1/2}$ for all $a \in A\}$. Then $\mathscr{L}$ is an ideal of A and for any $x \in \mathscr{L}$, $x^2 = 0$.*

Proof. Let $y_{1/2} \in \mathscr{L}$, $z_{1/2} \in A_{1/2}$, $x_1 \in A$. Clearly $(A_1 + A_0)(x_1y_{1/2}) + (x_1y_{1/2})(A_1 + A_0) \subseteq A_{1/2}$. Using (1)′ and (2)′ we find

$$(z_{1/2}, x_1, y_{1/2}) = e \circ (z_{1/2}, x_1, y_{1/2}) + z_{1/2} \circ (e, x_1, y_{1/2}), \tag{7}$$

$$(z_{1/2}, x_1, y_{1/2}) = e \circ (z_{1/2}, x_1, y_{1/2}) + y_{1/2} \circ (z_{1/2}, x_1, e). \tag{8}$$

Thus $z_{1/2} \circ (e, x_1, y_{1/2}) = y_{1/2} \circ (z_{1/2}, x_1, e) \in A_{1/2}$. Using (7) we then have $z_{1/2} \circ (e, x_1, y_{1/2}) = 0$ and then $(z_{1/2}, x_1, y_{1/2}) \in A_{1/2}$. Hence $z_{1/2}(x_1y_{1/2}) \in A_{1/2}$. Interchanging $z_{1/2}$ and $y_{1/2}$ we find that $(y_{1/2}x_1)z_{1/2} \in A_{1/2}$. Setting $z = x_1$, $y = y_{1/2}$, $w = z_{1/2}$, $x = e$ in (2)′ we find $(x_1, y_{1/2}, z_{1/2}) = e \circ (x_1, y_{1/2}, z_{1/2})$ (since $(x_1, y_{1/2}, e) = 0$). Thus $(x_1y_{1/2})z_{1/2} \in A_{1/2}$ for $y_{1/2}z_{1/2} \in A_{1/2}$. A similar substitution in (1)′ yields $z_{1/2}(y_{1/2}x_1) \in A_{1/2}$ so that $x_1y_{1/2}, y_{1/2}x_1 \in \mathscr{L}$. Replacing x_1 by x_0 we also find $x_0y_{1/2}, y_{1/2}x_0 \in \mathscr{L}$.

Next we consider $y_{1/2} \in \mathscr{L}$, $z_{1/2}$, $x_{1/2} \in A_{1/2}$. Substituting in (1)′ we obtain $(y_{1/2}, x_{1/2}, z_{1/2}) = e \circ (y_{1/2}, x_{1/2}, z_{1/2}) + y_{1/2} \circ (e, x_{1/2}, z_{1/2})$. Since $y_{1/2} \in \mathscr{L}$, $y_{1/2} \circ (e, x_{1/2}, z_{1/2}) \in A_{1/2}$. Thus we must have $(y_{1/2}, z_{1/2}, x_{1/2})_i = 0$; $i = 0,1$. Hence $(y_{1/2}x_{1/2})z_{1/2} \in A_{1/2}$ and, using *Theorem* 3 (and 2′),

$$(y_{1/2}x_{1/2})z_{1/2} = -(x_{1/2}y_{1/2})z_{1/2}, \quad z_{1/2}(x_{1/2}y_{1/2}) = -z_{1/2}(y_{1/2}x_{1/2}) \in A_{1/2}.$$

Thus $x_{1/2}y_{1/2} = -y_{1/2}x_{1/2} \in \mathscr{L}$ and $\mathscr{L}$ must be an ideal of A with the property that $\mathscr{L} \subseteq A_{1/2}$. Therefore $x^2 = 0$ for all $x \in \mathscr{L}$.

We next show that A with the added condition that A possess no ideals $\mathscr{L}$ such that $x^2 = 0$ for all $x \in \mathscr{L}$ must have a Peirce decomposition.

THEOREM 5. *Suppose A has no ideals $\mathscr{L} \neq 0$ such that $x^2 = 0$ for all $x \in \mathscr{L}$. Then for e an idempotent of A we have $A = A_{11} + A_{10} + A_{01} + A_{00}$, where $x \in A_{ij}$ if and only if $ex = ix$, $xe = jx$.*

Proof. It is well known that a necessary and sufficient condition that the decomposition of the theorem holds in A is that

$$(x, e, e) = (e, x, e) = (e, e, x) = 0 \text{ for all } x \in A.$$

Since we already have $(e, x, e) = 0$ we can reduce the proof to showing that $(x, e, e) = (e, e, x)$ for $x \in A_{1/2}$. If $x \in A_{1/2}$ we have

$$e(xe) = (e,e,x) = -(x,e,e) = (ex)e.$$

By the previous theorem we see that it suffices to show that $e(xe) \in \mathscr{L}$, $\mathscr{L}$ the ideal defined in *Theorem* 4. This result follows from the next *lemma*.

LEMMA. *Let A be a ring with idempotent e, and suppose $x_{1/2}, y_{1/2} \in A_{1/2}$. Then*

$$(x_{1/2}y_{1/2})_1 = [(ex_{1/2})(y_{1/2}e)]_1, \quad (x_{1/2}y_{1/2})_0 = [(x_{1/2}e)(ey_{1/2})]_0;$$

$$(ex_{1/2})(ey_{1/2}), (x_{1/2}e)(y_{1/2}e) \in A_{1/2}.$$

Proof. Identities (1) and (2) yield

$$(e, x_{1/2}, y_{1/2}) = e \circ (e, x_{1/2}, y_{1/2}), \quad (x_{1/2}, y_{1/2}, e) = e \circ (x_{1/2}, y_{1/2}, e).$$

Hence $(e, x_{1/2}, y_{1/2})_1 = (e, x_{1/2}, y_{1/2})_0 = 0$ and $(x_{1/2}, y_{1/2}, e)_1 = (x_{1/2}, y_{1/2}, e)_0 = 0$ so that

$$[(ex_{1/2})y_{1/2}]_1 = (x_{1/2}y_{1/2})_1, \quad [(ex_{1/2})y_{1/2}]_0 = 0,$$

$$[x_{1/2}(y_{1/2}e)]_1 = (x_{1/2}y_{1/2})_1, \quad [x_{1/2}(y_{1/2}e)]_0 = 0.$$

The lemma is immediate after we note that $ex_{1/2} + x_{1/2}e = x_{1/2}$.

At this juncture we are able to show that under the hypothesis of *Theorem* 5 the A_{ij} satisfy the same multiplicative relations as in the alternative case and this we proceed to do.

Since $(x_{11}, y_{1/2}, e) = 0$ we have

$$(x_{11}, y_{10}, e) = (x_{11}y_{10})e = 0, \quad (x_{11}, y_{01}, e) = (x_{11}y_{01})e - x_{11}y_{01} = 0.$$

Using the substitution $w = e$, $x = x_{11}$, $y = e$, $z = y_{01}$ in (1)$'$ results in

$$2(x_{11}, e, y_{01}) = e \circ (x_{11}, e, y_{01}) + x_{11} \circ (e, e, y_{01})$$

or

$$2x_{11}y_{01} = e(x_{11}y_{01}) + (x_{11}y_{01}e) = e \circ (x_{11}y_{01}).$$

But $x_{11}y_{01} \in A_{10} + A_{01}$ (by *Theorem* 3) so that $x_{11}y_{01} = e \circ (x_{11}y_{01}) = 2x_{11}y_{01}$. Hence $x_{11}y_{01} = 0$. Another application of (1) yields $(e, x_{11}, y_{10}) = e \circ (e, x_{11}, y_{10})$ or

$$x_{11}y_{10} - e(x_{11}y_{10}) = e(x_{11}y_{10}) - e(e(x_{11}y_{10})) + (x_{11}y_{10})e - (e(x_{11}y_{10}))e.$$

But the right-hand member is 0 since $(x_{11}y_{10})e = 0$. Thus, $x_{11}y_{10} \in A_{10}, x_{11}y_{01} = 0$ and using (2) and (2)$'$ we obtain $y_{01}x_{11} \in A_{01}, y_{10}x_{11} = 0$. Replacing x_{11} by x_{00} we find the corresponding relations $x_{00}y_{10} = y_{01}x_{00} = 0$, $x_{00}y_{01} \in A_{01}$, $y_{10}x_{00} \in A_{10}$.

Let $x_{10}, y_{10} \in A_{10}$. Then (1)$'$ yields

$$(x_{10}, e, y_{10}) = e \circ (x_{10}, e, y_{10}) + x_{10} \circ (e, e, y_{10})$$

or

$$x_{10}y_{10} = e \circ (x_{10}, e, y_{10}) = e(x_{10}y_{10}) + (x_{10}y_{10})e.$$

Thus $x_{10}y_{10} \in A_{10} + A_{01}$. Using (3)$'$ we find $x_{10}^2 e = e x_{10}^2$. Therefore $x_{10}^2 = 0$ and we have $x_{10}y_{10} = -y_{10}x_{10} \in A_{10} + A_{01}$. In a similar manner we find $x_{01}^2 = 0$, $y_{01}x_{01} = -x_{01}y_{01} \in A_{10} + A_{01}$.

Next suppose $x_{10} \in A_{10}$, $y_{01} \in A_{01}$. Then (1)$'$ becomes

$$(x_{10}, y_{01}, e) = e \circ (x_{10}, y_{01}, e) + x_{10} \circ (e, y_{01}, e)$$

or

$$(x_{10}y_{01})e - x_{10}y_{01} = e(x_{10}y_{01})e - e(x_{10}y_{01}).$$

Hence $x_{10}y_{01} \in A_{11} + A_{10} + A_{01}$ and interchanging x_{10} and y_{01} we find

$$(y_{01}x_{10})e = e(y_{01}x_{10})e + (y_{01}x_{10})e$$

so that $e(y_{01}x_{10})e = 0$ and $y_{01}x_{10} \in A_{10} + A_{01} + A_{00}$.

From the relation $ex^2 = x^2 e$ for all $x \in A_{10} + A_{01}$ we see that $x_{10} \circ y_{01} \in A_1 + A_0$.

Finally we show that $(x_{10}y_{01})_{10}$, $(x_{10}y_{01})_{01}$, $(x_{10}y_{10})_{10}$, $(x_{01}y_{01})_{01}$ belong to the ideal $\mathscr{L}$ of *Theorem* 4, and hence must be zero. In order to get $(x_{10}y_{01})_{10} \in \mathscr{L}$ it suffices to prove that $(x_{10}y_{01})_{10}z_{01}$, $z_{01}(x_{10}y_{01})_{10} \in A_{10} + A_{01}$. Identity (1)$'$ implies that

$$(x_{10}, y_{01}, z_{01}) = e \circ (x_{10}, y_{01}, z_{01}) + x_{10} \circ (e, y_{01}, z_{01})$$

while (2)$'$ yields

$$(x_{10}, y_{01}, z_{01}) = e \circ (x_{10}, y_{01}, z_{01}) + z_{01} \circ (x_{10}, y_{01}, e).$$

Combining these two relations we have

$$x_{10} \circ (e, y_{01}, z_{01}) = z_{01} \circ (x_{10}, y_{01}, e).$$

But $(e, y_{01}, z_{01}) \in A_{10}$ so that the left member is zero. Hence,

$$z_{01} \circ (x_{10}, y_{01}, e) = z_{01} \circ (x_{10}y_{01})_{10} = 0.$$

Therefore $[z_{01}(x_{10}y_{01})_{10}]_0 = [(x_{10}y_{01})_{10}z_{01}]_1 = 0$, and $(x_{10}y_{01})_{10} \in \mathscr{L}$. Replacing z_{01} by z_{10} in the foregoing results in $(x_{10}y_{01})_{01} \in \mathscr{L}$. The first relation above implies that $(x_{10}, y_{01}, z_{01})_i = 0$ for $i = 0, 1$. Thus $[(x_{10}y_{01})_{10}z_{01}]_1 = [x_{10}(y_{01}z_{01})_{01}]_1$. But the left member is zero since $(x_{10}y_{01})_{10} \in \mathscr{L}$ so that $(y_{01}z_{01})_{01} \in \mathscr{L}$. In a similar manner we see that $(x_{10}y_{10})_{10} \in \mathscr{L}$. Combining these remarks we have

THEOREM 6. *Suppose A satisfies the hypothesis of Theorem 5. Then for any idempotent e of A, $A = A_{11} + A_{10} + A_{01} + A_{00}$ where $A_{ij}A_{km} = \delta_{jk}A_{im}$ except when $i \neq j$ and $i = k$, $j = m$ and then $A_{ij}^2 \subseteq A_{ji}$.*

In the remainder of this section we suppose that A satisfies the hypothesis of *Theorem* 5.

THEOREM 7. *$A_{10}A_{01} + A_{10} + A_{01} + A_{01}A_{10}$ is an ideal of A.*

Proof. For the proof we need only show that $A_{10}A_{01}$ and $A_{01}A_{10}$ are ideals of A_{11} and A_{00} respectively. Let $x_{11} \in A_{11}$, $y_{10} \in A_{10}$, $z_{01} \in A_{01}$. Then (2)′ implies that $(x_{11}, y_{10}, z_{01}) = e \circ (x_{11}, y_{10}, z_{01}) \in A_{10} + A_{01}$. But $(x_{11}, y_{10}, z_{01}) \in A_{11}$ so that $(x_{11}y_{10})z_{01} = x_{11}(y_{10}z_{01})$ or $A_{11}(A_{10}A_{01}) \subseteq A_{10}A_{01}$. In a similar fashion we see that $(A_{10}A_{01})A_{11} \subseteq A_{10}A_{01}$ and, interchanging 1's and 0's we have the corresponding results for $A_{01}A_{10}$.

COROLLARY 1. *$A_{10}A_{01}$ and $A_{01}A_{10}$ are associative subrings of A.*

Proof. Using the proof of the preceding theorem we see that

$$\begin{aligned}(x_{11}(y_{10}z_{01}))w_{11} &= ((x_{11}y_{10})z_{01})w_{11} = (x_{11}y_{10})(z_{01}w_{11})\\ &= x_{11}(y_{10}(z_{01}w_{11})) = x_{11}((y_{10}z_{01})w_{11}).\end{aligned}$$

Since every element of $A_{10}A_{01}$ is the sum of elements of the form $y_{10}z_{01}$ we have established the associativity of $A_{10}A_{01}$. The same proof works for $A_{01}A_{10}$ as soon as we interchange 1's and 0's.

COROLLARY 2. *If A is simple then either $e = 1$ or $A_{11} = A_{10}A_{01}$ and $A_{00} = A_{01}A_{10}$.*

We are now in a position to state our main result.

THEOREM 8. *Let A be a simple ring satisfying (1), (2), and (3). Suppose A has an idempotent $e \neq 1$. Then A is either an associative ring or a Cayley-Dickson algebra over its center.*

Proof. A ring is alternative if and only if

$$(x, y, z) = \varepsilon(\sigma)(\sigma(x), \sigma(y), \sigma(z)) \tag{9}$$

for all permutations σ where $\varepsilon(\sigma) = 1$ or -1 as σ is even or odd. We prove the theorem by showing that (9) holds for all possible choices of x, y, z belonging to the A_{ij} since then Albert's result is applicable [2].

Combining Corollaries 1 and 2 of *Theorem* 7 we have $(x_{ii}, y_{ii}, z_{ii}) = 0$, $i = 0,1$. Suppose $x_{11}, y_{11} \in A_{11}, z_{10} \in A_{10}$. Then we see that $(z_{10}, x_{11}, y_{11}) = (z_{10}, y_{11}, x_{11}) = (x_{11}, z_{10}, y_{11}) = (y_{11}, z_{10}, x_{11}) = 0$. Next using (1)′ we have $2(x_{11}, y_{11}, z_{10}) = e \circ (x_{11}, y_{11}, z_{10}) + x_{11} \circ (e, y_{11}, z_{10}) = (x_{11}, y_{11}, z_{10}) \in A_{10}$. Thus $(x_{11}, y_{11}, z_{10}) = (y_{11}, x_{11}, z_{10}) = 0$. Replacing z_{10} by $z_{01} \in A_{01}$ we find the corresponding result. Clearly $(x_{11}, y_{11}, z_{00}) = (x_{11}, z_{00}, y_{11}) = (z_{00}, x_{11}, y_{11}) = 0$ for $z_{00} \in A_{00}$. Now suppose we examine products involving $x_{11} \in A_{11}$, $y_{10}, z_{10} \in A_{10}$. If we substitute $w = e$, $x = x_{11}$, $y = y_{10}$, $z = z_{10}$ in (1)′ we obtain

$$\begin{aligned}2(x_{11}, y_{10}, z_{10}) &= e \circ (x_{11}, y_{10}, z_{10}) + x_{11} \circ (e, y_{10}, z_{10}),\\ (x_{11}y_{10})z_{10} &= (y_{10}z_{10})x_{11}.\end{aligned}$$

Then using the fact that $a_{10}b_{10} = -b_{10}a_{10}$ we find $(x_{11}y_{10})z_{10} = -z_{10}(x_{11}\,y_{10}) = (y_{10}z_{10})x_{11} = -(z_{10}y_{10})x_{11} = -(x_{11}z_{10})y_{10} = y_{10}(x_{11}z_{10})$. Combining these we have $(x_{11}, y_{10}, z_{10}) = \varepsilon(\sigma)(\sigma(x_{11}), \sigma(y_{10}), \sigma(z_{10}))$ for all σ. Again, replacing y_{10}, z_{10} by y_{01}, x_{01} we have the corresponding results. The case $x_{11} \in A_{11}$, $y_{10} \in A_{10}$, $z_{01} \in A_{01}$ was done in the proof of *Theorem* 7 as soon as we note that $(y_{10}, x_{11}, z_{01}) = 0$ and $(z_{01}, x_{11}, y_{10}) = 0$ by setting $x = e$, $w = z_{01}$, $y = x_{11}$, $z = y_{10}$ in (1)′. If we replace x_{11} by x_{00} the corresponding results are proved in the same fashion.

We have reduced the proof to considering $x, y, z \in A_{10} + A_{01}$. First suppose that $x_{10}, y_{10}, z_{10} \in A_{10}$. Then (1)′ implies that $(x_{10}, y_{10}, z_{10}) = e \circ (x_{10}, y_{10}, z_{10}) + x_{10} \circ (e, y_{10}, z_{10})$. Equating the A_{00}-components we obtain

$$(x_{10}y_{10})z_{10} = (y_{10}z_{10})x_{10}.$$

A similar substitution in (2)′ yields

$$x_{10}(y_{10}z_{10}) = z_{10}(x_{10}y_{10}).$$

Then noting that $a_{10}b_{10} = -b_{10}a_{10}$ we see that $(x_{10}, y_{10}, z_{10}) = \varepsilon(\sigma)(\sigma(x_{10}), \sigma(y_{10}), \sigma(z_{10}))$ for all σ. The case $x_{01}, y_{01}, z_{01} \in A_{01}$ is proved in the same way. Finally we consider $x_{10}, z_{10} \in A_{10}$, $y_{01} \in A_{01}$. Then $(y_{01}, x_{10}, z_{10}) = -(y_{01}, z_{10}, x_{10}) = (x_{10}, z_{10}, y_{01}) = -(z_{10}, x_{10}, y_{01})$ since $y_{01}(x_{10}z_{10}) = -y_{01}(z_{10}x_{10}) = (z_{10}x_{10})y_{01} = -(x_{10}z_{10})y_{01}$. Consider $(x_{10}, y_{01}, z_{10}) + (y_{01}, x_{10}, z_{10}) = w_{10} \in A_{10}$. We show that $x_{01}w_{10} = w_{10}x_{01} = 0$ for all $x_{01} \in A_{01}$. Then $Aw_{10} + w_{10}A \subseteq A_{10} + A_{01}$ so that w_{10} belongs to the ideal $\mathscr{L}$ of *Theorem* 3 and hence, must be zero.

$$\begin{aligned} x_{01}w_{10} &= x_{01}(x_{10}, y_{01}, z_{10}) - x_{01}(y_{01}(x_{10}z_{10})) \\ &= x_{01}(x_{10}, y_{01}, z_{10}) - (x_{10}z_{10})(x_{01}y_{01}). \end{aligned}$$

Since $x_{10}z_{10} \in A_{01}$ and $a_{01}(b_{01}c_{01}) = c_{01}(a_{01}b_{01})$.
Setting $x = x_{01}$, $w = x_{10}$, $y = y_{01}$, $z = z_{10}$ in (1)′ yields

$$0 = (x_{01} \circ x_{10}, y_{01}, z_{10}) = x_{01} \circ (x_{10}, y_{01}, z_{10}) + x_{10} \circ (x_{01}, y_{01}, z_{10}) .$$

Since the A_{00}-component of the right member must be zero we have

$$\begin{aligned} 0 &= x_{01}(x_{10}, y_{01}, z_{10}) + (x_{01}, y_{01}, z_{10})x_{10} \\ &= x_{01}(x_{10}, y_{01}, z_{10}) + [(x_{01}y_{01})z_{10}]x_{10} \\ &= x_{01}(x_{10}, y_{01}, z_{10}) + (z_{10}x_{10})(x_{01}y_{01}) \\ &= x_{01}(x_{10}, y_{01}, z_{10}) - (x_{10}z_{10})(x_{01}y_{01}) \\ &= x_{01}w_{10} . \end{aligned}$$

In a similar fashion we have $w_{10}x_{01} = 0$. Hence, from our preceding remarks $w_{10} = 0$, so that $(x_{10}, y_{01}, z_{10}) = -(y_{01}, x_{10}, z_{10})$. Interchanging x_{10} and z_{10} we obtain $(z_{10}, y_{01}, x_{10}) = -(y_{01}, z_{10}, x_{10})$. Combining these results we have $(x_{10}, y_{01}, z_{10}) = \varepsilon(\sigma)(\sigma(x_{10}), \sigma(y_{01}), \sigma(z_{10}))$ for all σ. Replacing x_{10}, z_{10}, y_{01} by x_{01}, z_{01}, y_{10} we obtain $(x_{01}, y_{10}, z_{01}) = \varepsilon(\sigma)(\sigma(x_{01}), \sigma(y_{10}), \sigma(z_{01}))$ and the theorem is proved. See §5 for an example to show this result is *not* valid for simple rings without idempotent $e \neq 1$.

4. **Semi-simple algebras.** Let A be a finite-dimensional algebra over field F satisfying (1), (2), (3). We define the radical N of A to be the maximal nil ideal of A. This makes sense since A is power-associative by *Theorem* 1. A is said to be semi-simple if $N = 0 \neq A$.

THEOREM 9. *Let e be a principal idempotent of A. Then $A_{1/2} + A_0 \subseteq N$, N the nil radical of A.*

THEOREM 10. *Let A be semi-simple algebra satisfying* (1), (2), *and* (3). *Then A has a unity element and is the direct sum of simple algebras.*

Proof. The proofs of these theorems are the same as those of the corresponding results given in [4] and we do not repeat them here.

5. **Examples.** We begin with

EXAMPLE 1. Let A be any Lie ring. Then, since $x^2 = 0$ and $x \circ y = 0$ for all $x, y \in A$, the identities (1), (2), and (3) must hold in A. Hence, there are simple finite-dimensional nil algebras satisfying (1), (2), and (3) (the simple Lie algebras), so that postulating the existence of an idempotent severely limits the possibilities for A when A is simple.

EXAMPLE 2. In [4] we defined a construction which gave rise to a class of simple finite-dimensional algebras satisfying the identity $(x, y, z) = (z, y, x)$, in which the flexible identity $(x, y, x) = 0$ fails. Hence, these algebras (which possess unity elements) cannot be alternative. A direct calculation shows that the algebra A of this class which is given by the basis $\{1, x, y\}$ where $x^2 = y^2 = 0, xy = -yx = 1$ satisfies (1), (2), and (3). Thus, *Theorem* 8 is in this sense the best possible result.

EXAMPLE 3. Let A be an algebra over the field F with a basis $\{e, x, y\}$ where $e^2 = e, ex = x + y, xe = -y, ey = y, ye = x^2 = y^2 = xy = yx = 0$. We see that $A_1 = Fe, A_{1/2} = Fx + Fy, A_0 = 0$. If $z = \alpha e + \beta x + \gamma y, \alpha, \beta, \gamma \in F$ then $z^2 = \alpha z$ so that A is power-associative and satisfies (3). Any easy calculation reveals that $(w, u, v) \in A_{1/2}$ for all $w, u, v \in A$. But then $(z^2, u, v) = (\alpha z, u, v) = \alpha(z, u, v)$ while $z \circ (z, u, v) = \alpha e \circ (z, u, v) = \alpha(z, u, v)$. Hence, $(z^2, u, v) = z \circ (z, u, v)$ and (1) holds. In a similar fashion (2) must be valid in A. We see that $e(xe) = (ex)e = -y \neq 0$ so that $A_{1/2}$ does not decompose into $A_{10} + A_{01}$. Therefore *Theorem* 5 is nontrivial.

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